Homotopical methods in complex geometry

The surprising power of topology

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✨ whirlwind tour ✨

Overview

(If things go to plan)

1. **Why topology might seem useless**
2. **Some counter-examples**
   A. *Simplicial connections*
   B. *Stein and Oka manifolds*
   C. *Deligne cohomology*
   D. *Twisting cochains, deformation theory, and analytic HAG*
3. **Yet another way to define vector bundles**

1. Why topology might seem useless

Being continuous is "easy"

- Every complex-manifold has an underlying topological space.
- Every holomorphic map is continuous (and even infinitely differentiable).
- Every holomorphic vector bundle has an underlying smooth vector bundle.
- ...
- In general, there is a forgetful functor from "holomorphic structure" to "topological structure".

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Being holomorphic is “hard”

- If a holomorphic map is constant on any open neighbourhood then it is globally constant.
- If a function is holomorphic on the whole of $\mathbb{C}$, then it is either constant, or such that $f(\mathbb{C})$ is either the whole of $\mathbb{C}$ or $\mathbb{C}\setminus\{a\}$ for some $a \in \mathbb{C}$.
- If a holomorphic function has an essential singularity, then, on any punctured neighbourhood of this singularity, the function attains all values in $\mathbb{C}$, with at most a single exception, infinitely often.
- …

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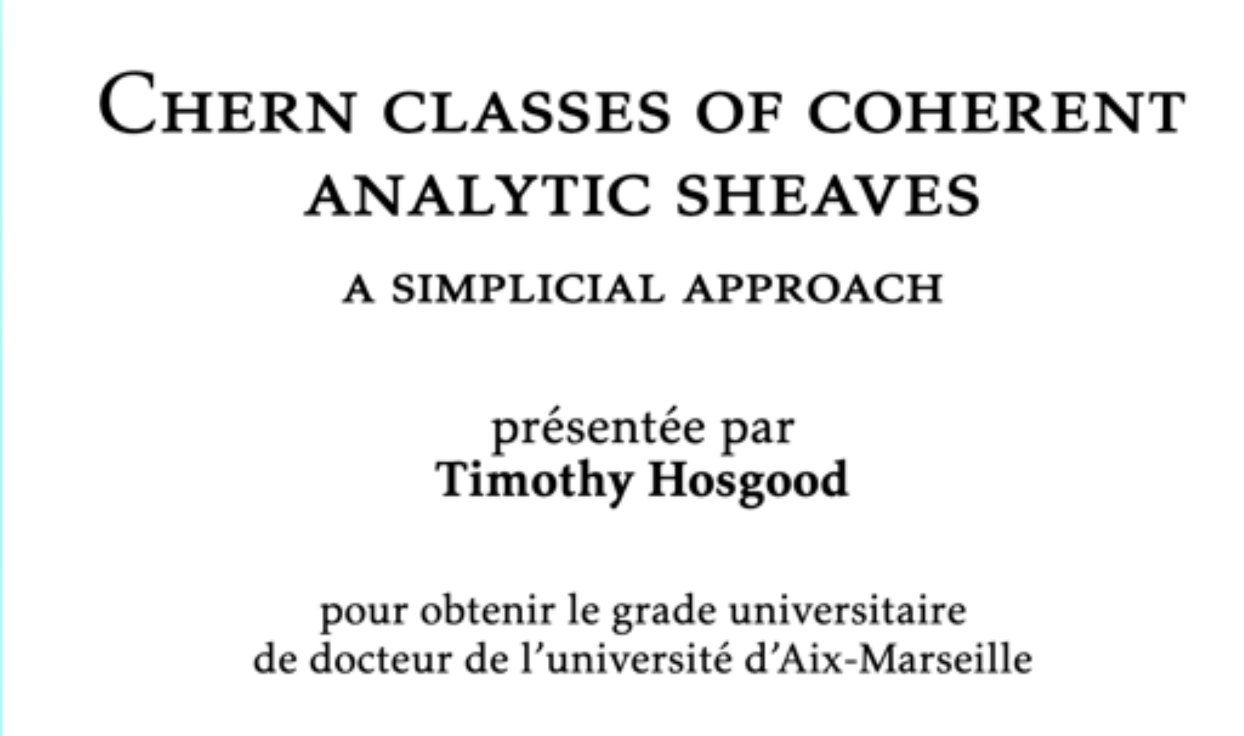
2. Some counter-examples

Simplicial connections

- Given a vector bundle $E \to X$, a *connection* on E is a linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying the Leibniz rule: $\nabla(fs) = \mathrm{d}f \otimes s + f\,\nabla s$.
- **Fact.** Every smooth vector bundle admits a connection.
- **Fact.** Most holomorphic vector bundles do **not** admit a (global) connection.
- **Lemma.** If we replace X by the *Čech nerve* of X (a cofibrant replacement), and consider “simplicial connections”, then every holomorphic vector bundle admits a global “connection”.
- **Corollary.** We can use Chern–Weil theory to calculate Chern classes (i.e. characteristic classes).


Chern classes of coherent analytic sheaves

A simplicial approach

présentée par
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pour obtenir le grade universitaire
de docteur de l’université d’Aix-Marseille

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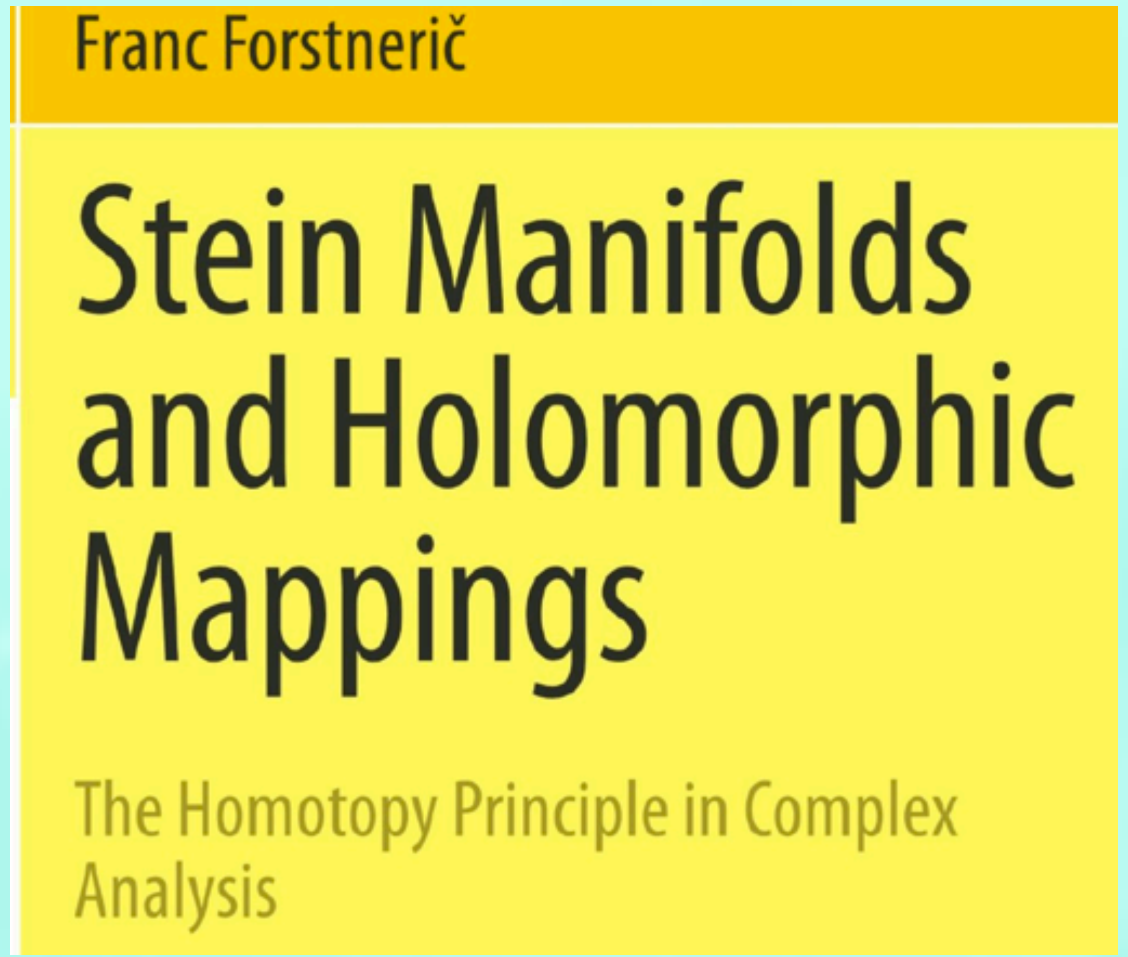
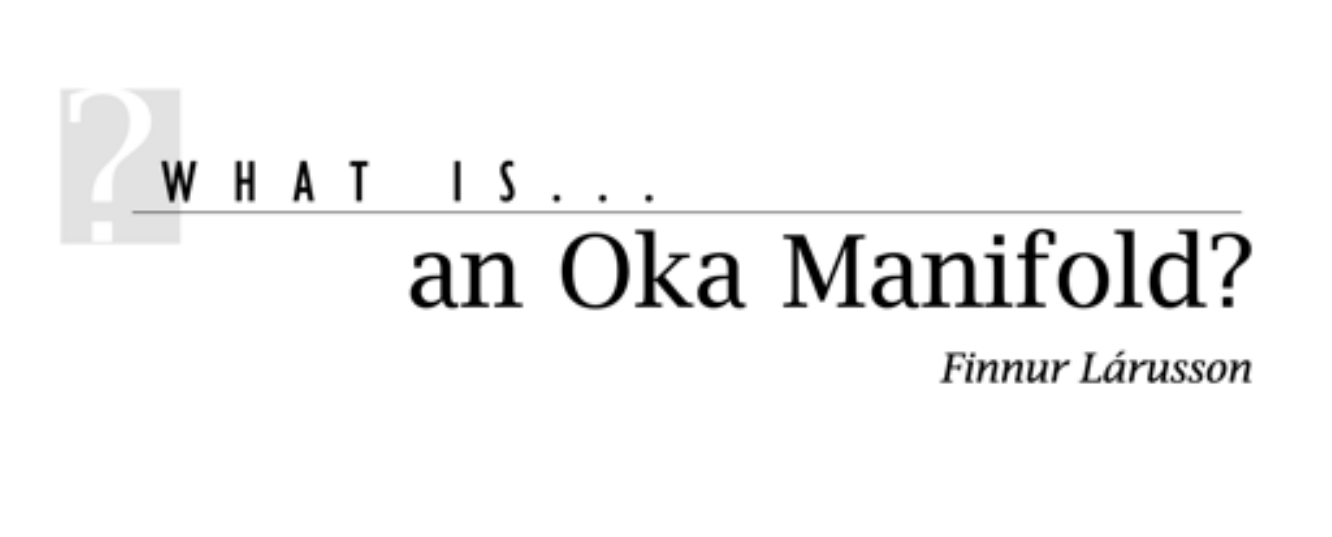
Model categories, and Stein and Oka manifolds

- A good holomorphic analogue to being contractible (or even being affine) is being *Stein*: holomorphically convex and holomorphically separable.
    - An equivalent definition: there are "lots" of holomorphic maps $X \to \mathbb{C}$ (enough to embed X as a closed complex sub-manifold of $\mathbb{C}$)
- Dual to this is the notion of being *Oka*: having "lots" of holomorphic maps $\mathbb{C} \to X$.
- Formally, there is a model category which contains all complex manifolds, and with Stein manifolds being cofibrant and Oka manifolds fibrant (with so-called *Oka maps* being fibrations).
- The *h-principle* (or *Oka principle* in complex geometry): when solutions to an *analytic* problem exist in the absence of *topological* obstructions.





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Deligne cohomology

HOLOMORPHIC GERBES AND THE BEILINSON REGULATOR
by Jean-Luc BRYLINSKI

GEOMETRY OF DELIGNE COHOMOLOGY
PAWEŁ GAJER

- Very well understood in the smooth case, but much less so in the holomorphic case.
- The *Deligne complex* is given by the homotopy pullback of a truncated de Rham complex and $\mathbb{Z}$.
- More concretely, $\mathbb{Z}_D(p) = (2\pi i)^p \mathbb{Z} \hookrightarrow \Omega^0 \to \Omega^1 \to \ldots \to \Omega^{p-1}$, which is quasi-isomorphic to $\mathcal{O}^\times \xrightarrow{\mathrm{d}\log} \Omega^1 \to \Omega^2 \to \ldots \to \Omega^{p-1}$.
- Sheaf cohomology of this complex classifies holomorphic connections on holomorphic line bundles (and, in higher degrees, holomorphic connective structures on holomorphic gerbes)

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Twisting cochains and deformation theory

- Well-known correspondence:

$$\text{locally free sheaves} \longleftrightarrow \text{vector bundles} \longleftrightarrow \text{principal } \mathrm{GL}_n\text{-bundles}$$

- Another well-known correspondence (in *algebraic* geometry (on a Noetherian affine scheme)):

$$\text{l.f. sheaves} \longleftrightarrow \text{f.g. proj. modules}$$

$$\text{coherent sheaves} \longleftrightarrow \text{f.g. modules}$$

$$\text{quasi-coherent sheaves} \longleftrightarrow \text{modules}$$

- Can we extend the first correspondence to the other two cases of the second correspondence? . . . Probably (maybe).

- The Maurer–Cartan equation, simplicial twisting cochains, etc., plus analytic HAG/DAG.

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3. Yet another way to define vector bundles

- Take a Lie group G, and consider the presheaf Y_G on the category Man of smooth manifolds given by the Yoneda embedding: $Y_G = C^\infty(-, G)$.
- We can endow this with the structure of a Lie group, thanks to the Lie group structure of G (i.e. "pointwise-ly"), which gives us a presheaf Y_G of Lie groups.
- We can deloop Y_G to obtain a presheaf $\mathbb{B}Y_G$ of one-object groupoids, i.e. for any smooth manifold M, we have the groupoid $\mathbb{B}Y_G(M)$ with one object $*$ and with automorphisms $\mathrm{Hom}(*, *) \cong \mathsf{Man}(M, G)$.
- We can take the nerve of this to obtain a presheaf $\mathscr{N}\mathbb{B}Y_G$ of simplicial sets.
- Abstractly, we've built a functor $\mathscr{N}\mathbb{B}Y\colon \mathsf{LieGroup} \to [\mathsf{Man}^{\mathrm{op}}, \mathsf{sSet}]$.
- We can pull this back along the (opposite of) the Čech nerve $\check{\mathscr{N}}\colon \mathsf{Man}_{\mathscr{U}} \to [\Delta^{\mathrm{op}}, \mathsf{Man}]$ to get a functor $(\check{\mathscr{N}}^{\mathrm{op}})^*\mathscr{N}\mathbb{B}Y\colon \mathsf{LieGroup} \to [\mathsf{Man}_{\mathscr{U}}^{\mathrm{op}}, \mathsf{csSet}]$.
- Finally, we can apply totalisation to this, and obtain $\mathrm{Tot}((\check{\mathscr{N}}^{\mathrm{op}})^*\mathscr{N}\mathbb{B}Y)\colon \mathsf{LieGroup} \to [\mathsf{Man}_{\mathscr{U}}^{\mathrm{op}}, \mathsf{sSet}]$.

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- **Lemma.** $\mathrm{Tot}((\check{\mathcal{N}}^{\mathrm{op}})^*\mathcal{N}\mathbb{B}Y_{\mathrm{GL}_n(\mathbb{C})})(X,\mathcal{U})$ classifies complex vector bundles on X in a precise way:
  - its π_0 consists of isomorphism classes of $\mathrm{GL}_n(\mathbb{C})$-principal bundles on X;
  - its π_1 based at some isomorphism class $[E]$ is the gauge group of E;
  - all higher π_i are zero.

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